Recitation 6

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Review

An $m \times n$ matrix A defines $A\colon \mathbb{R}^n \to \mathbb{R}^m$. The following are equivalent:

- A is **onto**;
- columns of A span $\mathbb{R}^m$;
- system $Ax = b$ has a solution (is consistent) for all b;
- **every row** of A has a pivotal position;

Once again, TFAE:

- A is **one-to-one**;
- $Ax = Ay$ implies that $x = y$ (i.e. A maps different vectors to different vectors);
- $Ax = 0$ has only the trivial solution (i.e. A doesn't kill any non-zero vectors);
- columns of A are linearly independent;
- **every column** of A is pivotal.

Basis: vectors $v_1, \ldots, v_n$ in a vector space V for a **basis** if and only if

- $v_1, \ldots, v_n$ are independent;
- $v_1, \ldots, v_n$ span the whole V.

Eigenvalue: a number λ is an **eigenvalue** of a matrix A if $Av = \lambda v$ for some **non-zero vector** v.
Eigenvector: vector v as above is called an **eigenvector** corresponding to the eigenvalue λ.

Stochastic matrix: a matrix with non-negative entries, numbers in each column should add up to 1.
Probability vector x is a vector with non-negative entries, numbers should add up to 1.
Equilibrium vector: for a stochastic matrix P is a probability vector q s.t. $Pq = q$.

Problems

Problem 1. Let's consider the following hypothetical situation. Suppose one TA is grading his students prelim solutions. After reading a solution, he is either happy, or sad and frustrated. If the TA is happy now, then there is 90% chance he won't get frustrated after reading one more solution. If the TA is already sad, there is a 20% chance he will cheer up after one more solution (the chance is not too big because of a beer shortage in the area, mostly in the area of his office, which basically leaves the TA without any source of happiness).

1. What is the stochastic matrix for this situation?
2. If the TA is fine now, what is the probability he will lose it after reading two more solutions? three more solutions?
3. Suppose you know that there is a 30% chance the TA is sad now. What is the chance he will be happy after grading the next exam?
4. There are A LOT of students, so the grading takes eternity. Compute what percentage of time the TA will be spending frustrated *eventually*.

Problem 2. Suppose vectors $\{v_1, v_2, v_3\}$ in $\mathbb{R}^5$ are linearly independent. Prove that the vectors $\{v_1 + 3v_2 - v_3, -v_1 - 2v_2 + v_3, v_2 + 2v_3\}$ are also linearly independent.

Problem 3. Is $\lambda = 2$ an eigenvalue of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$?

Problem 4. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Is $\lambda = 4$ and eigenvalue? $\lambda = -1$? $\lambda = 1$? $\lambda = 3$?
For each number which turned out to be an eigenvalue, find an eigenvector corresponding to this eigenvalue.

Problem 5. For the matrix

$$A = \begin{bmatrix} 1 & 3 & -3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

find all eigenvalues and eigenvectors corresponding to them. Why are the eigenvector you found linearly independent?

Problem 6. Suppose you know that a matrix A has 0 as an eigenvalue. Can A be invertible?

Problem 7. Suppose A is an invertible $n \times n$ matrix, and you know its eigenvalues $\lambda_1 \ldots, \lambda_n$. What are the eigenvalues of A^{-1}? (Hint: use the definition of eigenvalue and the previous problem.)

Problem 8. Look at the definition of the equilibrium vector q for a stochastic matrix P. Does it have anything to do with eigenvectors and eigenvalues?

Problem 9. Mark each of the following statements as true or false:

- If $v_1.v_2$ are two linearly independent vectors, they mus correspond to distinct eigenvalues.
- The eigenvalues of a matrix are on its diagonal.
- The null space of a matrix A is spanned by some of eigenvectors.
- To find the eigenvalues of A you need to reduce A to the echelon form.

Problem 10. Without calculations, find an eigenvalue of the matrix of the following linear transformations, and describe the corresponding eigenspace.

- $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ reflecting with respect to the line $x = y$.
- $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ rotating around the y-axis by the angle $\pi/2$.

Find matrices of the above transformations and check your answers.

Problem 11. Matrix A is called nilpotent if $A^n = 0$ for some number n. Give an example of a 3×3 non-zero nilpotent matrix. What are the eigenvalues of any nilpotent matrix A?